1 Images

1.1 Finding an equation for an image

To find the equation, you must first setup $Ax = b$.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1.1}$$

Augment these two matrices together to form

$$[A|b] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & x \\ 2 & 0 & 2 & y \\ 0 & 1 & 1 & z \end{array}\right] \tag{1.2}$$

Row reduce this to upper-triangular form. Wherever a row on the A side of the matrix contains only 0s, the b side of the matrix must equal 0 for the system to be possible.

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & x \\ 0 & 2 & 2 & y - 2x \\ 0 & 0 & 0 & z - (y/2) + x \end{array}\right] \tag{1.3}$$

The only way there can be a solution is if $z - (y/2) + x = 0$ as otherwise the equation is impossible. Multiply this equation by 2 to simplify it and get matrix C.

$$C = \left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \end{array}\right] \tag{1.4}$$

1.2 Testing for Img membership

To test if b_1 is a member of Img(A), or, if $Ax = b_1$ is possible, $c \cdot b_1 = 0$ must be true. c is the transverse of matrix C for the Img(a).

1.3 Parameterizing images

Row reduce C as in $Cb = 0$. For each y in C that does not have a unique solution, replace with t_x.

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \end{array}\right] \rightarrow \begin{array}{c} y_1 - y_2 + t_1 + 0 = 0 \\ 0 - y_2 + 2t_1 + t_2 = 0 \end{array} \rightarrow \begin{array}{c} y_1 = y_2 - t_1 \\ y_2 = 2t_1 + t_2 \end{array} \tag{1.5}$$

Then, the values of y_x just need to be loaded into a matrix such that:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ 2t_1 + t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{1.6}$$

2 Closest Vector

We want to find the closest vector (least-squared) of $Ax = b$. To get this, find the transpose of A, A^t. Multiply both sides of the equation by A^t to get $A^tAx = A^tb$. Solve for x by mutiplying boths sides by $(A^tA)^{-1}$. b is a vector containing your output, A is the system turned into a matrix and x is a vector of your coefficients.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}, b = \begin{bmatrix} 1.9 \\ 3.7 \\ 6.2 \\ 7.7 \\ 10.5 \end{bmatrix} \text{ and } x = \begin{bmatrix} a \\ b \end{bmatrix} \tag{2.1}$$

The matrix A expresses $v_i = a + bU_i$ for $i = 1 \ldots 5$. The vector b contains the measured outputs for $i1 \ldots 5$ and x is a placeholder vector showing that we have 2 degrees of flexibility in finding a solution. We then calculate A^tA and A^tb and augment them together. Solve this augmented matrix to get the values for a and b.

$$A^tA = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \text{and } A^tb = \begin{bmatrix} 30.0 \\ 111.2 \end{bmatrix} \tag{2.2}$$

$$\left[\begin{array}{cc|c} 5 & 15 & 30.0 \\ 15 & 55 & 111.2 \end{array}\right] \tag{2.3}$$

$$a = \frac{-9}{25} \text{ and } b = \frac{53}{25} \tag{2.4}$$

If you wish to find how accurate or "noisy" your data is, you can compare your least-squared solution to the actual solution by calculating the distance $|c - b|$ where b is our least-squared vector and c is the theoretical output, or $c = Ax_0$.

2.1 Computing the distance to the Img(a)

Let's find the vector m that is actually possible as output of Img(a) that is closest to our data, b. First calculate c, or Ax. This gives us a vector that is the output of Img(a) using our manufactured coefficients, or by using our generated line equation to run the numbers. This output, c, has been munged from the actual output to match the line. Take the actual output we used originally, b, and subtract c. Then take the magnitude of this vector to get the distance between the two, d. We can then solve for r which is the ratio between our difference and the original output b. This is how much change to the data b is required to make the system solvable.

$$d = \left|\left|\vec{b} - \vec{c}\right|\right| \tag{2.5}$$

$$r = \frac{x}{\left|\left|\vec{b}\right|\right|} \tag{2.6}$$

2.2 Computing the distance between two lines

To do this, we must subtract line L_2 from line L_1 and then form this into a matrix. If L_1 and L_2 are expressed parametrically, we have:

$$\begin{array}{l} L_1 = x_1 + sV_1 \\ L_2 = x_2 + tV_2 \end{array} \rightarrow L_1 - L_2 = sV_1 - tV_2 - (x_2 - x_1) \tag{2.7}$$

$$L_1 - L_2 = \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} - \begin{bmatrix} x_2 - x_1 \end{bmatrix} \rightarrow L_1 - L_2 = As - b \rightarrow As = b \quad (2.8)$$

This now just becomes a least-squared problem solving for s. Plug the two values of s back into L_1 and L_2 and find the magnitude of $L_1 - L_2$ to get the distance between the two lines which are closest when the parameters are equal to matrix s.

3 Subspaces

A subspace is just a set of vectors that can be combined and still be in the same subspace.

3.1 Spanning Sets

A set of vectors spans a subspace if the whole subspace can be regenerated by that set. Ex: Subspace $s = t_1\vec{V_1} + t_2\vec{V_2} + \cdots + t_n\vec{V_n}$, where vectors $\vec{V_n}$ are all members of the spanning set.

3.2 Linear Independence

A set of vectors is linearly independant if when row reduced there are no non-pivotable variables.

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad (3.1)$$

Row reduce this to get

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.2)$$

This has a non-pivotal variable, and therefore this set is not linerally independant.

3.3 Basis

A vector set is a basis for a subspace if it spans it and is linerally independant. A basis is basically the simpliest set that still can generate the subspace.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (3.3)$$

(3.3) is an example of a basis for the subspace that is all real 3D vectors.

4 Orthogonal Complements

The orthogonal complement of a subspace is a set of vectors that are all perpendicular to the subspace, or, $v \cdot x = 0$ for every v in the subspace.

4.1 Orthogonal Projection

An orthogonal projection P of A onto subspace s can be expressed as

$$P_s = A(A^t A)^{-1} \tag{4.1}$$

4.2 Finding an orthonormal basis

Start with a basis

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Now, normalize all vectors in the base, or, $u_n = \frac{v_n}{|v_n|}$. The first vector in the set, w_1, is u_1. The projection of vectors $v_n \ldots v_1$ is done by working from w_1 up to w_n with $w_2 = v_2 - (v_2 \cdot u_1)u_1$. The general form can be expressed as

$$w_n = v_n - \sum_{i=1}^{n-1} (v_n \cdot u_i)u_i \tag{4.2}$$