Calculus I, II and a bit of III

Kurt Nelson

November 11, 2009

1 Calculus Identities

1.1 Derivatives

$$\frac{d}{dx}|u| = \frac{u}{|u|}u' \tag{1.1}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}u' \tag{1.2}$$

$$\frac{d}{dx}e^u = e^u u' \tag{1.3}$$

$$\frac{d}{dx}\log_a u = \frac{1}{u * \ln a}\, u' \tag{1.4}$$

$$\frac{d}{dx}a^u = \ln a * a^u\, u' \tag{1.5}$$

$$\frac{d}{dx}\sin u = \cos u\, u' \tag{1.6}$$

$$\frac{d}{dx}\cos u = -\sin u\, u' \tag{1.7}$$

$$\frac{d}{dx}\tan u = \sec^2 u\, u' \tag{1.8}$$

$$\frac{d}{dx}\cot u = -\csc^2 u\, u' \tag{1.9}$$

$$\frac{d}{dx}\sec u = \sec u \tan u\, u' \tag{1.10}$$

$$\frac{d}{dx}\csc u = -\csc u - \cot u\, u' \tag{1.11}$$

$$\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}} \tag{1.12}$$

$$\frac{d}{dx}\arccos u = \frac{-u'}{\sqrt{1-u^2}} \tag{1.13}$$

$$\frac{d}{dx}\arctan u = \frac{u'}{1+u^2} \tag{1.14}$$

$$\frac{d}{dx}\mathrm{arc}\cot u = \frac{-u'}{1+u^2} \tag{1.15}$$

$$\frac{d}{dx}\operatorname{arc\,sec} u = \frac{u'}{|u|\sqrt{u^2-1}} \tag{1.16}$$

$$\frac{d}{dx}\operatorname{arc\,csc} u = \frac{-u'}{|u|\sqrt{u^2-1}} \tag{1.17}$$

1.2 Integrals

$$\int a^u\,du = \frac{1}{\ln a}a^u + C \tag{1.18}$$

$$\int e^u\,du = e^u + C \tag{1.19}$$

$$\int \sin u\,du = -\cos u + C \tag{1.20}$$

$$\int \cos u\,du = \sin u + C \tag{1.21}$$

$$\int \tan u\,du = -\ln|\cos u| + C \tag{1.22}$$

$$\int \cot u\,du = \ln\sin u + C \tag{1.23}$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \tag{1.24}$$

$$\int \csc u\,du = -\ln|\csc u + \cot u| + C \tag{1.25}$$

$$\int \sec^2 u\,du = \tan u + C \tag{1.26}$$

$$\int \csc^2 u\,du = -\cot u + C \tag{1.27}$$

$$\int \sec u \tan u\,du = \sec u + C \tag{1.28}$$

$$\int \csc u \cot u\,du = -\csc u + C \tag{1.29}$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C \tag{1.30}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C \tag{1.31}$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C \tag{1.32}$$

2 Trigonometry

2.1 Substitutions

Form	Substitution
$\sqrt{a^2-u^2}$	$u = a\cdot\sin\theta$
$\sqrt{a^2+u^2}$	$u = a\cdot\tan\theta$
$\sqrt{u^2-a^2}$	$u = a\cdot\sec\theta$

2.2 Identities

2.2.1 Pythagorean

$$\sin^2 x + \cos^2 x = 1 \tag{2.1}$$

$$1 + \tan^2 x = \sec^2 x \tag{2.2}$$

$$1 + \cot^2 x = \csc^2 x \tag{2.3}$$

2.2.2 Cofunction

$$\sin \frac{\pi}{2} - x = \cos x \sec \frac{\pi}{2} - x = \csc x \tan \frac{\pi}{2} - x = \cot x \tag{2.4}$$

2.2.3 Sum and Difference

$$\sin u \pm v = \sin u \cos v \pm \cos u \sin v \tag{2.5}$$

$$\cos u \pm v = \cos u \cos v \mp \sin u \sin v \tag{2.6}$$

$$\tan u \pm v = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v} \tag{2.7}$$

2.2.4 Double-Angle

$$\sin 2u = 2 \sin u \cos u \tag{2.8}$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u \tag{2.9}$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u} \tag{2.10}$$

2.2.5 Product to Sum

$$\sin u \sin v = \frac{1}{2}\left[\cos u - v - \cos u + v\right] \tag{2.11}$$

$$\cos u \cos v = \frac{1}{2}\left[\cos u - v + \cos u + v\right] \tag{2.12}$$

$$\sin u \cos v = \frac{1}{2}\left[\sin u + v + \sin u - v\right] \tag{2.13}$$

$$\cos u \sin v = \frac{1}{2}\left[\sin u + v - \sin u - v\right] \tag{2.14}$$

2.2.6 Power Reducing

$$\sin^2 u = \frac{1 - \cos 2u}{2} \tag{2.15}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2} \tag{2.16}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u} \tag{2.17}$$

3 Methods

3.1 Taking Integrals

3.1.1 Improper Integrals

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx = \lim_{\beta\to\infty}\int_{\beta}^{c} f(x)dx + \lim_{\alpha\to\infty}\int_{c}^{\alpha} f(x)dx \tag{3.1}$$

If limit exists, the integral converges. There may be infinite discontinuity: $\int_0^1 \frac{dx}{x}; x = 0, \infty$

3.1.2 Euler's Method

$$y' = F(x, y); x_0, y_0 \tag{3.2}$$

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \tag{3.3}$$

3.1.3 Logistic Differential Equations

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right); \frac{dy}{dt} > 0, y < L \uparrow; \frac{dy}{dt} < 0, y < L \downarrow \tag{3.4}$$

$$y = \frac{L}{1 + be^{-kt}}; L = \text{carrying capacity}, k = \text{constant}, b = \frac{1}{c} \tag{3.5}$$

2 asymptotes: Maximum and 0

3.1.4 First-order Linear Differential Equations

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{3.6}$$

$$y = \frac{1}{e^{\int P(x)dx}} \int Q(x)e^{\int P(x)dx} + C \tag{3.7}$$

3.1.5 Bernoulli Equation

$$y' + P(x)y = Q(x)y^n \tag{3.8}$$

$$y^{1-n}e^{\int (1-n)P(x)dx} = \int (1-n)Q(x)e^{\int (1-n)P(x)dx}dx + C \tag{3.9}$$

3.2 Functions

3.2.1 Homogeneous Function

$$f(tx, ty) = t^n f(x, y)\text{Homogeneous function of degree n}\text{Change of variables:}y = vx; dy = v\,dx + x\,dv \tag{3.10}$$

3.2.2 Exponential Growth and Decay

$$y = Ce^{kt}; C = \text{initial value of y}, k = \text{proportionality constant} \tag{3.11}$$

If $k > 0$, it is growth. If $k < 0$, it is decay.

4 Conics

4.1 Circle

$$(x-h)^2 + (y-k)^2 = r^2 \tag{4.1}$$

4.2 Parabola

If the open side of the parabola is along the vertical axis:

$$(x-h)^2 = 4p(y-k) \tag{4.2}$$

If the open side of the parabola is along the horizontal axis:

$$(y-k)^2 = 4p(x-h) \tag{4.3}$$

4.3 Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{4.4}$$

$$c^2 = a^2 - b^2 \tag{4.5}$$

$$e = \frac{c}{a} \tag{4.6}$$

As e approaches 1, the elipse becomes a circle

4.4 Hyperbola

Horizontal Transverse Axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{4.7}$$

Vertical Transverse Axis

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{4.8}$$

$$c^2 = a^2 + b^2 \tag{4.9}$$

$$e = \frac{c}{a} > 1 \tag{4.10}$$

Asymptotes

$$y = k \pm \frac{b}{a}(x-h) \tag{4.11}$$

5 Parametrics

5.1 Derivative

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{5.1}$$

5.2 Arclength

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \tag{5.2}$$

5.3 Surface Area

Convert

$$\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned} \tag{5.3}$$

About x-axis

$$A = 2\pi \int_a^b g(t)\sqrt{f'(t)^2 + g'(t)^2} dt \tag{5.4}$$

About y-axis

$$A = 2\pi \int_a^b f(t)\sqrt{f'(t)^2 + g'(t)^2} dt \tag{5.5}$$

6 Polar

6.1 Definitions

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{6.1}$$

$$x^2 + y^2 = r^2 \tag{6.2}$$

$$\tan\theta = \frac{y}{x} \tag{6.3}$$

6.2 Distance

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\left(\theta_1 - \theta_2\right)} \tag{6.4}$$

6.3 Arclength

$$s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \tag{6.5}$$

6.4 Surface Area

About polar axis

$$A = 2\pi \int_\alpha^\beta r\sin\theta\sqrt{r^2 + (r')^2} d\theta \tag{6.6}$$

About line $\theta = \frac{\pi}{2}$

$$A = 2\pi \int_\alpha^\beta r\cos\theta\sqrt{r^2 + (r')^2} d\theta \tag{6.7}$$

6.5 Area

$$A = \frac{1}{2} \int_\alpha^\beta r^2 d\theta \tag{6.8}$$

6.6 Conic Section

e= eccentricity and d = distance from pole to directrix

$$r = \frac{ed}{1 \pm e\sin\theta} \tag{6.9}$$

$$r = \frac{ed}{1 \pm e\cos\theta} \tag{6.10}$$

7 Solids and Surfaces

7.1 Disk Method

$$V = \pi \int_a^b R(x)^2 dx \tag{7.1}$$

7.2 Washer Method

$$V = \pi \int_a^b R(x)^2 - r(x)^2 dx \tag{7.2}$$

7.3 Shell Method

$$V = 2\pi \int_a^b P(x)h(x) dx \tag{7.3}$$

7.4 Arc Length

$$s = \int_a^b \sqrt{1 + f'(x)^2} dx \tag{7.4}$$

7.5 Surface Area

$$A = 2\pi \int_a^b r(x)\sqrt{1 + f'(x)^2} dx \tag{7.5}$$